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Extensions of Pontryagin Hypergroups

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(joint work with H. Heyer)

Abstract

The purpose of this note is to investigate the extension problem for the category of commutative hypergroups. In fact, by applying the new notion of a field of compact subhypergroups, sufficiently many extensions can be established, and among them splitting extensions can be characterized. Moreover, the duality of extensions will be studied via duality of fields of hypergroups. The method of extension via fields of hypergroups yields the construction of Pontryagin hypergroups which do not arise from group-theoretic objects.

1 Introduction

Let H and L be hypergroups. Then, a hypergroup K is called an extension of L by H if the sequence:

$$1 \to H \to K \to L \to 1$$

is exact. If the quotient hypergroup K/H is defined, this is equivalent to the fact that K/H is isomorphic to L. Here the notions of subhypergroup, quotient hypergroup and isomorphism between hypergroups are taken from [B-H], a source from which all the basic knowledge on hypergroups needed in the sequel will be taken.

There exist several methods to construct extensions of hypergroups from given ones. These methods lead to an insight into the structure of hypergroups. One of the methods is based on the notion of hypergroup join as introduced by Jewett [J] and further developed by Dunkl-Ramirez[D-R], Fournier-Ross[F-R], Voit[V_2], Vrem[Vr_1], and Zeuner[Z]. The join $H \vee L$ of a compact commutative hypergroup H and a discrete commutative hypergroup L can be interpreted as the minimal extension of L by H. On the other hand the maximal extension of L by H is the product hypergroup $H \times L$. The purpose of the present discussion is to construct by generalizing the method of join sufficiently many extensions which in some sense are larger than the join and smaller than the product. The method of substitution introduced by Voit [V_2] is another generalization of the join which provides extensions of hypergroups. The relation between the construction presented in this work and the substitution will be clarified in section 6.

In the course of the paper for two commutative hypergroups H and L such that each connected component of L is an open set, we shall give the definition of a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ of compact subhypergroups $H(\ell)$ of H based on L, and show that every field φ gives rise to an extension $K(H, \varphi, L)$ of L by H as described in Theorem 3.1. Moreover, for strong hypergroups H and L such that each connected component of both L and the dual $\hat{H}$ of H is an open set, we shall introduce the dual $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$ of the field φ and show in Theorem 4.4 that the extension $K(\hat{L}, \hat{\varphi}, \hat{H})$ of $\hat{H}$ by $\hat{L}$ is isomorphic to the dual of $K(H, \varphi, L)$. The latter property implies that if both H and L are Pontryagin hypergroups, then $K(H, \varphi, L)$ is also a Pontryagin hypergroup. By applying the method of fields one can also obtain Pontryagin hypergroups not arising from group-theoretic objects as for example orbital actions and Gelfand pairs. This new aspect is illustrated in Examples 7.2 and 7.3.

In order to investigate the structure of hypergroups it will be essential to determine all extensions K of L by H for given commutative hypergroups H and L. In the corresponding discussion we give a characterization of extensions obtained by a field of compact subhypergroups. Those extensions will be called splitting extensions. If L is a discrete commutative hypergroup, it will be shown in Theorem 5.1 that all splitting extensions of L by H are determined by the construction via fields of compact subhypergroups. It is known that in general there are extensions which do not split. It remains still an open problem to determine all extensions of commutative hypergroups, a problem that waits for a solution.

2 Preliminaries

In this section we recapitulate the principal notions from the basic theory of hypergroups by stressing those definitions and properties which are essential in the course of the discussion. We start with the definition of a hypergroup along the axiomatics established by Dunkl, Jewett, and Spector. Further elements of the theory can be taken from the monograph [B-H].

Let K be a locally compact (Hausdorff) space. We write $C(K)$ for the space of continuous complex-valued functions on K. The space $C(K)$ has various distinguished subspaces, $C_b(K)$, $C_0(K)$, and $C_c(K)$, the spaces of bounded continuous functions, those that vanish at infinity, and those with compact support respectively. Both $C_b(K)$ and $C_0(K)$ are topologized by the uniform norm $\|\cdot\|_\infty$. We denote by $M_b(K)$, $M_b^+(K)$ and $M^1(K)$ the spaces of bounded measures, non-negative bounded measures and probability measures on K respectively. For each $\mu \in M_b(K)$ the support of μ is denoted by $\mathrm{supp}(\mu)$ and the norm of μ is given by $\|\mu\| := \sup\{|\mu(f)| : f \in C_c(K), \|f\|_\infty \leq 1\}$. The symbol ε_x stands for the Dirac measure at $x \in K$. By $\mathcal{C}(K)$ we denote the space of non-empty compact subsets of K, furnished with the Michael-Hausdorff topology.

Definition A *hypergroup* $K := (K, *)$ consists of a locally compact space together with an associative product (called convolution) $*$ on $M_b(K)$ satisfying the following conditions:

(1) The space $M_b(K)$ admits a convolution $*$ and an involution $^-$ such that $(M_b(K), *, ^-)$ is an involutive Banach algebra with respect to the norm $\|\cdot\|$.

(2) The mapping $(\mu, \nu) \longmapsto \mu * \nu$ from $M_b^+(K) \times M_b^+(K)$ into $M_b^+(K)$ is continuous with respect to the weak topology in $M_b(K)$.

(3) For $x, y \in K$ the convolution product $\varepsilon_x * \varepsilon_y$ belongs to $M^1(K)$ and $\mathrm{supp}(\varepsilon_x * \varepsilon_y)$ is compact.

(4) The mapping $K \times K \ni (x, y) \longmapsto \mathrm{supp}(\varepsilon_x * \varepsilon_y) \in \mathcal{C}(K)$ is continuous.

(5) There exisits a unit element e of K such that $\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$ for all $x \in K$.

(6) There exists an involutive homeomorphism $x \mapsto x^-$ in K such that $(\varepsilon_x * \varepsilon_y)^- = \varepsilon_y^- * \varepsilon_x^-$ and $e \in \mathrm{supp}(\varepsilon_x * \varepsilon_y)$ if and only if $x = y^-$ for all $x, y \in K$.

A hypergroup K is said to be *commutative* if the convolution $*$ in $M_b(K)$ is commutative, and *hermitian* if the involution $^-$ is the identity mapping. There are prominent classes of commutative hypergroups arising from orbital actions and Gelfand pairs, and also large classes of examples constructed on $\mathbb{Z}_+$ and $\mathbb{R}_+$ by polynomial and Sturm-Liouville methods respectively. The reader is encouraged to check the details in [B-H].

For subsets A and B of K one defines

$$A * B = \bigcup_{x \in A, y \in B} \mathrm{supp}(\varepsilon_x * \varepsilon_y).$$

If $x \in K$, we write $x * A$ or $A * x$ instead of $\{x\} * A$ or $A * \{x\}$ respectively.

A non-empty closed subset H of K is called a *subhypergroup* if $H * H = H = H^-$, where $H^- = \{x \in K : x^- \in H\}$. A subhypergroup H is said to be *normal* if $x * H = H * x$, and *supernormal* if $x^- * H * x \subset H$ for all $x \in K$.

Let $(K,*)$ and $(L,\circ)$ be two hypergroups with units e_K and e_L respectively. A continuous mapping $\varphi : K \longrightarrow L$ is said to be a *hypergroup homomorphism* if $\varphi(e_K) = e_L$ and

$$\varepsilon_{\varphi(x)} \circ \varepsilon_{\varphi(y)} = \varphi(\varepsilon_x * \varepsilon_y)$$

whenever $x, y \in K$. A hypergroup homomorphism $\varphi : K \longrightarrow L$ is said to be an *isomorphism* if φ is a homeomorphism. If $\iota : H \longrightarrow K$ is an injective hypergroup homomorphism and $p : K \longrightarrow L$ is a surjective hypergroup homomorphism such that $\iota(H) = p^{-1}(L)$, one says that the sequence

$$1 \to H \to K \to L \to 1$$

is *exact* and that K is an *extension* of L by H. We note that the quotient K/H does not necessarily have a hypergroup structure in this situation.

Here we shall recall some facts on quotient hypergroups. Let $p : K \longrightarrow L$ be an open and surjective hypergroup homomorphism. Then $H := p^{-1}(L)$ is a normal subhypergroup of K, $K/H := \{x * H : x \in K\}$ is a locally compact space with respect to the quotient topology, and the formula

$$\varepsilon_{x*H} * \varepsilon_{y*H} := \int_K \varepsilon_{z*H}(\varepsilon_x * \varepsilon_y)(dz) \qquad (*)$$

for all $x, y \in K$ defines a hypergroup structure on K/H such that K/H is isomorphic to L, where $(*)$ is understood as an equality of linear functionals on $C_c(K/H)$. Conversely, if H is a normal subhypergroup of K such that $(*)$ defines a hypergroup structure, then the mapping $x \longmapsto x * H$ from K onto K/H is an open hypergroup homomorphism. This statement is especially available if H is a compact normal subhypergroup. Moreover, if H is supernormal in K or a closed subgroup in K or if H is contained in a compact subgroup in K, then K/H is always a hypergroup. For details see [R] and [Vr_2].

Next we shall review the notion of substitution introduced by Voit in [V_2]. Let H and M be hypergroups and $\pi : H \longrightarrow M$ be a proper and open hypergroup homomorphism. We put $Q := \pi(H) \subset M$ and $L := M/Q$. Then Voit in [V_2] established a hypergroup $S(M, Q \to H) := (H \cup (M \setminus Q), *)$ by substituting the open subhypergroup Q in M to H via π which is an extension of L by H. It is clear that the hypergroup join $H \vee L$ of a compact hypergroup H and a discrete hypergroup L coincides with the substitution $S(L, \{e_L\} \to H)$ when the unit e_L of L is replaced by H and $\pi : H \longrightarrow \{e_L\} \subset L$ is the trivial hypergroup homomorphism. Both the substitution and the join will serve as motivating examples for the extensions to be discussed in this work.

Now we shall describe some facts from the duality theory of commutative hypergroups. Let K be a commutative hypergroup. For a Borel measurable function f on K and $x, y \in K$ we write

$$f(x * y) := \int_K f(z) d(\varepsilon_x * \varepsilon_y)(z)$$

if this integral exists. For each $x \in K$ the translation T^x on such functions f and on measures μ is defined by

$$(T^x f)(y) = f(x * y) \ (y \in K) \quad \text{and} \quad (T^x \mu)(f) = \mu(T^x f).$$

A measure $\omega \neq 0$ is called a *Haar measure* of K if it satisfies that $T^x\omega = \omega$ for all $x \in K$. It is known that every commutative hypergroup K has a Haar measure ω_K which is unique up to a positive multiplicative constant. If K is compact, ω_K is finite and hence can be normalized to become a probability measure.

A complex-valued function χ on K is called a *character* of K if χ is a bounded continuous function on K satisfying

$$\chi(e) = 1, \quad \chi(x * y) = \chi(x)\chi(y), \text{ and } \chi(x^-) = \overline{\chi(x)}$$

for all $x, y \in K$. The set $\hat{K}$ of all characters of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets. One calls $\hat{K}$ the *dual* of K. In general the dual $\hat{K}$ is not necesarily a hypergroup. If $(\hat{K}, \hat{*})$ becomes a hypergroup with respect to a convolution $\hat{*}$ which is defined by the product of characters on K, then K is said to be a *strong* hypergroup. In this case $\hat{\hat{K}} := \widehat{(\hat{K})}$ is also defined as a locally compact space. If $\hat{\hat{K}}$ is a hypergroup and is isomorphic to K, then K is called a *Pontryagin* hypergroup.

3 Fields of compact subhypergroups

Let $H = (H, *)$ and $L = (L, *)$ be commutative hypergroups with units e_H and e_L respectively. We assume that each connected component of L is an open set.

Definition A family $\{H(\ell) : \ell \in L\}$ of subsets of H will be called a *field of compact subhypergroups of H based on L* and denoted by $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ if it satisfies the following conditions :

(1) Each $H(\ell)$ is a compact subhypergroup of H with $H(e_L) = \{e_H\}$ and $H(\ell^-) = H(\ell) \quad (\ell \in L)$.

(2) For ℓ_1, ℓ_2, and $\ell \in L$ such that $\ell \in \text{supp}(\varepsilon_{\ell_1} * \varepsilon_{\ell_2})$ we have $[H(\ell_1) * H(\ell_2)] \supset H(\ell)$, where $[H(\ell_1) * H(\ell_2)]$ is the closed hypergroup generated by $H(\ell_1)$ and $H(\ell_2)$.

(3) For ℓ_1 and ℓ_2 contained in a connected component of L, $H(\ell_1) = H(\ell_2)$ holds.

Let $\omega(\ell)$ denote the normalized Haar measure of $H(\ell)$. Then condition (2) is equivalent to

(4) $\omega(\ell_1) * \omega(\ell_2) = \omega(\ell_1) * \omega(\ell_2) * \omega(\ell)$ whenever $\ell \in \text{supp}(\varepsilon_{\ell_1} * \varepsilon_{\ell_2})$.

Now let $Q(\ell)$ denote the quotient hypergroup $H/H(\ell)$, and let K denote the disjoint union of the hypergroups $Q(\ell)$ ($\ell \in L$), i.e.

$$K := \bigcup_{\ell \in L} Q(\ell) = \{(h * H(\ell), \ell) \ : \ h \in H, \ell \in L\}.$$

The topology of K is induced by the canonical mapping

$$\pi \ : H \times L \ni (h, \ell) \longmapsto (h * H(\ell), \ell) \in K.$$

It is easy to deduce from conditions (1) to (3) that K is a locally compact space. The Dirac measure of an element $(h * H(\ell), \ell) \in K$ is given as the measure

$$(\varepsilon_h * \omega(\ell) \otimes \varepsilon_\ell \in M_b(H) \otimes M_b(L),$$

and the convolution $*_\varphi$ in $M_b(H) \otimes M_b(L)$ is well-defined by

$$((\varepsilon_{h_1} * \omega(\ell_1)) \otimes \varepsilon_{\ell_1}) *_\varphi ((\varepsilon_{h_2} * \omega(\ell_2)) \otimes \varepsilon_{\ell_2}) = (\varepsilon_{h_1} * \varepsilon_{h_2} * \omega(\ell_1) * \omega(\ell_2)) \otimes \varepsilon_{\ell_1} * \varepsilon_{\ell_2}.$$

The set K together with the convolution $*_\varphi$ associated with the field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ will be denoted by $K(H, \varphi, L)$. We get the following

Theorem 3.1. Let H and L be commutative hypergroups such that every connected component of L is an open set, and let $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ be a field of compact subhypergroups of H based on L. Then $K(H, \varphi, L)$ is a commutative hypergroup and an extension of L by H.

4 Duality of fields and hypergroups

Let H and L be strong hypergroups such that every connected component of both L and the dual $\hat{H}$ of H is an open set, and let $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ be a field of compact subhypergroups of H based on L. Then for each $\ell \in L$ we choose $X(\ell)$ to be the annihilator $A(\hat{H}, H(\ell)) := \{\chi \in \hat{H} : \chi(x) = 1 \text{ for all } x \in H(\ell))\}$ of $H(\ell)$ in the dual $\hat{H}$ of H.

Next, for each $\chi \in \hat{H}$ set

$$Y(\chi) = \{\ell \in L : \chi \in X(\ell)\}.$$

Finally, for each $\chi \in \hat{H}$ we introduce

$$Z(\chi) = A(\hat{L}, Y(\chi))$$

and obtain the following

Proposition 4.1. The family $\{Z(\chi) \subset \hat{L} : \chi \in \hat{H}\}$ gives rise to a field $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$ of compact subhypergroups of $\hat{L}$ based on $\hat{H}$.

We call the field $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$ the *dual field* of $\varphi : L \ni \ell \longmapsto H(l) \subset H$. Associated with the dual field $\hat{\varphi}$ one can construct an extension $K(\hat{L}, \hat{\varphi}, \hat{H})$ of $\hat{H}$ by $\hat{L}$. We arrive at the following duality theorem.

Theorem 4.4. Let $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ be a field of compact subhypergroups of a strong hypergroup H based on a strong hypergroup L such that all connected components of L and $\hat{H}$ are open sets. Then

(1) $K(\hat{L}, \hat{\varphi}, \hat{H}) \cong \hat{K}(H, \varphi, L)$.

Moreover, if both H and L are Pontryagin hypergroups, then $K(H, \varphi, L)$ is also a Pontryagin hypergroup and

(2) $\hat{K}(\hat{L}, \hat{\varphi}, \hat{H}) \cong K(H, \varphi, L)$.

5 Splitting extensions

Let $H = (H, *)$ and $L = (L, \circ)$ be commutative hypergroups, and let K be an extension of L by H, i.e., the sequence

$$1 \to H \to K \to L \to 1$$

is exact. We say that the extension K of L by H *splits* or that K is a *splitting extension* if K satisfies the following conditions:

There exits a proper and continuous injective mapping ϕ from L into K such that

(1) $\phi(e_L) = e_K$ and $\phi(\ell^-) = \phi(\ell)^-$.

(2) The sets $H(\ell) = \{h \in H : \varepsilon_h * \varepsilon_{\phi(\ell)} = \varepsilon_{\phi(\ell)}\}$ are compact subhypergroups of H with $H(\ell^-) = H(\ell)$.

(3) $\varepsilon_{\phi(\ell_1)} * \varepsilon_{\phi(\ell_2)} = \phi(\varepsilon_{\ell_1} \circ \varepsilon_{\ell_2}) * \omega(\ell_1) * \omega(\ell_2)$ for ℓ_1 and $\ell_2 \in L$, where $\omega(\ell)$ denotes the normalized Haar measure of $H(\ell)$.

(4) $\omega(\ell_1) * \omega(\ell_2) * \omega(\ell) = \omega(\ell_1) * \omega(\ell_2)$ for ℓ_1, ℓ_2, and $\ell \in L$ such that $\ell \in \operatorname{supp}(\varepsilon_{\ell_1} \circ \varepsilon_{\ell_2})$.

(5) $K = \{h * \phi(\ell) : h \in H \text{ and } \ell \in L\}$ and $H \cap \phi(L) = \{e_K\}$.

The subsequent result provides a characterization of extensions associated with a field of hypergroups as splitting extensions.

Theorem 5.1. Let H and L be commutative hypergroups such that every connected component of L is an open set. Then the extension $K(H, \varphi, L)$ associated with a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ splits. Conversely, if L is a discrete commutative hypergroup, then all splitting extensions of L by H are obtained in this way.

6 Relationship between substitution and extensions

Let H be a compact commutative hypergroup, and let L be a discrete commutative hypergroup. Then the hypergroup join $H \vee L$ is canonically defined and appears as a typical extension of H by L. In [V2], Voit developed the notion of substitution as a generalization of the hypergroup join. From the point of view of extension of hypergroups one can reformulate the notion of substitution in the following way.

For two exact sequences

$$1 \to W \to H \to Q \to 1$$

and

$$1 \to Q \to M \to L \to 1$$

the substitution $K = S(M, Q \to H) = (H \cup (M \setminus Q), \circ)$ is defined. K is called the *hypergroup obtained by substitution Q in M by H via $\pi : H \to Q \subset M$* , and it satisfies the exact sequences

$$1 \to H \to K \to L \to 1$$

and

$$1 \to W \to K \to M \to 1.$$

This extension K of L by H strongly depends on M. Our method to construct extensions associated with a field is different from the notion of substitution. However, there is some relationship between substitution and extension as shown below.

Case 1. If M is given as $K(Q, \psi, L)$ for some field $\psi : L \ni \ell \longmapsto Q(\ell) \subset Q$, the associated field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ is canonically defined by $H(\ell) = \pi^{-1}(Q(\ell))$, and we see that

$$S(K(Q, \psi, L), Q \to H) = K(H, \varphi, L).$$

Case 2. For a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ of compact subhypergroups of H based on L, take the common compact subhypergroup W of $H(\ell)$ for all $\ell \in L$ except $\ell = e_L$, for example,

$$W = \bigcap_{\ell \in L \setminus \{e_L\}} H(\ell).$$

Setting $Q = H/W$ and $Q(\ell) = H(\ell)/W \subset Q$ we obtain a field $\psi : L \ni \ell \longmapsto Q(\ell) \subset Q$. In this case we can take M as $K(Q, \psi, L)$, and we see that

$$K(H, \varphi, L) = S(K(Q, \psi, L), Q \to H).$$

If for each $\ell \in L$ except for $\ell = e_L$, $H(\ell)$ is equal to the fixed compact subhypergroup W of H, then

$$K(H, \varphi, L) = S(Q \times L, Q \to H).$$

Remark Here we note the triviality of substitution. If $W = \{e_H\}$, we see that $Q = H$ and $S(M, Q \to H) = M$. This is the trivial substitution. For $k \in S(M, Q \to H)$ such that $k \notin H$,

$$H \cap \mathrm{supp}(\varepsilon_k * \varepsilon_{k^-}) \supset W$$

always holds. Therefore, if the condition

$$H \cap \mathrm{supp}(\varepsilon_k * \varepsilon_{k^-}) = \{e_H\}$$

holds for some $k \in S(M, Q \to H)$ with $k \notin H$, the substitution must be trivial.

If for an extension K of L by H the condition

$$H \cap \mathrm{supp}(\varepsilon_k * \varepsilon_{k^-}) = \{e_K\}$$

holds for some $k \in K$ with $k \notin H$, K does not arise from non-trivial substitution. Consequently, $K(H, \varphi, L)$ does not arise from non-trivial substitution if $H(\ell) = \{e_H\}$ for some $\ell \in L$ ($\ell \neq e_L$). We note that this situation often occurs as will be shown in the next section.

7 Applications and examples

In the category of commutative hypergroups there are only few Pontryagin hypergroups which are not of group-theoretic origin in the sense that they do not arise from orbital actions and Gelfand pairs. Applying the method of fields of hypergroups one can provide many new examples of Pontryagin hypergroups. These examples show the strength of the method of fields of hypergroups and indicate the possibility of further investigations on the structure of commutative hypergroups.

Before describing our examples we prepare some well-known simple facts.

Let A be the smallest non-trivial hypergroup with

$$A = \{\ell_0, \ell_1\}, \quad \ell_1^2 = p\ell_0 + (1-p)\ell_1,$$

where ℓ_0 is the unit, $0 < p \leq 1$, and we write $\ell_i \ell_j$ instead of $\varepsilon_{\ell_i} * \varepsilon_{\ell_j}$.

Let B be $\mathbb{Z}_2 \times \mathbb{Z}_2$, namely,

$$B = \{\ell_0, \ell_1, \ell_2, \ell_3\},$$

$$\ell_1^2 = \ell_2^2 = \ell_3^2 = \ell_0, \ \ell_1\ell_2 = \ell_3, \ \ell_1\ell_3 = \ell_2, \ \ell_2\ell_3 = \ell_1.$$

Let C denote the simplest compact hypergroup which is given as an orbital hypergroup of the one-dimensional torus $\mathbb{T}$ by the action of $\mathbb{Z}_2$, i.e.

$$C = ([-1, 1], *),$$

$$\varepsilon_{\cos\theta_1} * \varepsilon_{\cos\theta_2} = \frac{1}{2}\,\varepsilon_{\cos(\theta_1+\theta_2)} + \frac{1}{2}\,\varepsilon_{\cos(\theta_1-\theta_2)}.$$

Finally, let D denote the simplest discrete hypergroup which arises from a random walk on $\mathbb{Z}$, i.e.

$$D = \{\ell_0, \ell_1, \ell_2, \ldots, \ell_n, \ldots\},$$

$$\ell_m\ell_n = \frac{1}{2}\,\ell_{|m-n|} + \frac{1}{2}\,\ell_{m+n} \quad (m, n = 0, 1, 2, \ldots).$$

Here we note that A and B are self-dual and $\hat{D} \cong C, \hat{C} \cong D$. These facts imply that A, B, C, and D are all Pontryagin hypergroups.

For a natural number a, $D(a)$ and $F(a)$ denote the subhypergroups of D and C which are defined by

$$D(a) = \{\ell_{an} \ : \ n = 0, 1, 2, \ldots\}$$

and

$$F(a) = \{\cos\frac{2k\pi}{a} \ : \ k = 0, 1, 2, \ldots, a-1\}$$

respectively.

Observe that

$$F(a) = A(C, D(a)), \quad D(a) = A(D, F(a)).$$

We denote the quotient hypergroup $C/F(a)$ by $C(a)$ and write it

$$C(a) = ([\cos\frac{\pi}{a}\ ,\ 1], *).$$

Example 7.1. Let H be a compact Pontryagin hypergroup and let $L = A = \{\ell_0, \ell_1\}$. Take any closed subhypergroup W of H and denote H/W by Q. Then we obtain a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$, where $H(\ell_0) = \{e_H\}$ and $H(\ell_1) = W$. This field φ gives rise to an extension of L by H of the form

$$K(H, \varphi, L) = S(Q \times L, Q \to H).$$

If we choose $H = C$ and $W = F(a)$, we get the concrete model

$$K(a) = \{[-1, 1] \cup [\cos\frac{\pi}{a}, 1],\ *\}$$

with a parameter a from the set of natural numbers.

Example 7.2. Let W_1 and W_2 be two compact subhypergroups of a compact Pontryagin hypergroup H and let $L = B = \{\ell_0, \ell_1, \ell_2, \ell_3\}$. When we put

$$H(\ell_0) = \{e_H\},\ H(\ell_1) = W_1,\ H(\ell_2) = W_2,\ H(\ell_3) = [W_1 * W_2],$$

we obtain a field $\varphi\ :\ L \ni \ell \longmapsto H(\ell) \subset H$ and an extension $K(H, \varphi, L)$ of L by H. With the choice $H = C$ and $W_1 = F(a)$, $W_2 = F(b)$ we see that $[W_1 * W_2] = F(c)$ for a natural number c which is the least common multiple of a and b. Hence, we arrive at an extension $K = K(a, b)$ which is concretely represented as

$$K(a, b) = \left([-1, 1] \cup [\cos\frac{\pi}{a}, 1] \cup [\cos\frac{\pi}{b}, 1] \cup [\cos\frac{\pi}{c}, 1],\ *\right).$$

In a similar way one can get the extensions $K_n = K(H, \varphi_n, L_n)$ for $L_n = B \times B \times \cdots \times B$ and $K_\infty = K(H, \varphi_\infty, L_\infty)$ with $L_\infty = B \times B \times \cdots \times B \times \cdots$. We note that L_∞ is the inductive limit of the sequence $\{L_n\ :\ n = 1, 2, \dots\}$ and K_∞ is the inductive limit of the sequence $\{K_n\ :\ n = 1, 2, \dots\}$.

Example 7.3. Let W_1 and W_2 be two compact subhypergroups of a compact Pontryagin hypergroup H, and let $L = D = \{\ell_0, \ell_1, \ell_2, \dots, \ell_n, \dots\}$. Putting

$$H(\ell_0) = \{e_H\},\ H(\ell_1) = [W_1 * W_2],\ H(\ell_2) = W_1,$$

$$H(\ell_3) = W_2,\ H(\ell_4) = W_1,\ H(\ell_5) = [W_1 * W_2],\ H(\ell_6) = W_1 \cap W_2$$

and

$$H(\ell_n) = H(\ell_k) \quad (n \equiv k \pmod 6),\ n \neq 0 \text{ and } k = 1,2,3,4,5,6)$$

we obtain a field $\varphi : L \ni \ell \longmapsto H(\ell) \subset H$ and an extension $K(H, \varphi, L)$ of L by H. If $H = C$, $W_1 = F(a)$, and $W_2 = F(b)$, we see that as above $[W_1 * W_2] = F(c)$ for a natural number c which is the least common multiple of a and b, and $W_1 \cap W_2 = F(d)$ for a natural number d which is the greatest common divisor of a and b. Thus we have an extension $K = K(a, b)$, where a and b are natural numbers.

It is easy to see that the dual hypergroup of $K(a, b)$ can be concretely described by the dual field $\hat{\varphi} : \hat{H} \ni \chi \longmapsto Z(\chi) \subset \hat{L}$. We give the description in the case that $1 < d < a < b < c$.

$\hat{H} = \{\chi_0, \chi_1, \chi_2, \cdots, \chi_n, \cdots\} \cong D$ and $\hat{L} \cong C = ([-1, 1], *)$,

$Z(\chi_n) = F(1)$ for $n \equiv 0 \pmod c$,

$Z(\chi_n) = F(2)$ for $n \equiv 0 \pmod a$ except $n \equiv 0 \pmod b$,

$Z(\chi_n) = F(3)$ for $n \equiv 0 \pmod b$ except $n \equiv 0 \pmod a$,

$Z(\chi_n) = F(6)$ for $n \equiv 0 \pmod d$ except $n \equiv 0 \pmod a$ and $n \equiv 0 \pmod b$,

$Z(\chi_n) = \hat{L}$ for othewise n.

We list further properties of the Pontryagin hypergroup $K(a, b)$.

(1) $K(a_1, b_1) \cong K(a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

(2) $K(1, 1) \cong C \times D$.

(3) $K(a, a) = S(C(a) \times D, C(a) \to C)$.

(4) $K(a, b)$ is self-dual if and only if $a = 2$ and $b = 3$.

(5) For the greatest common divisor d of a and b,

$$K(a, b) = S(M(d), C(d) \to C) \text{ for } M(d) = K(C(d), \psi, D).$$

(6) If a and b are coprime, $K(a, b)$ does not arise from non-trivial substitution. This follows from the facts that $H(\ell_6) = F(1) = \{e_H\}$ and $H \cap \operatorname{supp}((\varepsilon_{e_H} \otimes \varepsilon_{\ell_6})^- *_\varphi (\varepsilon_{e_H} \otimes \varepsilon_{\ell_6})) = \{e_K\}$.

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